

Comparisons of Quadratic Mating Methods

Slices, Medusas and Clusters, Thurston Equivalence and Shared Matings

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A quadratic mating consists of a quadratic rational function, whose Julia sets contain two complementary components, each of which is the Julia set of a quadratic polynomial located on complementary hemispheres of the Riemann sphere. Two approaches can be used. In the first, one of the rational function's critical points is given a fixed period and the other is allowed to vary, forming a parameter plane of matings. In the second, including Medusa, a sequence of coefficients is combinatorially generated from external angles, using theory of Thurston and others developed by John Hubbard. Included are observations of Julia set matings utilizing Medusa (Boyd & Henriksen 2012) and $\text{Per}_k(0)$ moduli space slices (Devaney et al. 2013) in Dark Heart (King 2016) and $\text{Per}_3(0)$ in Mandel (Jung 2014), also including Chéritat's (2015) and Sharland's (2012) mating examples.

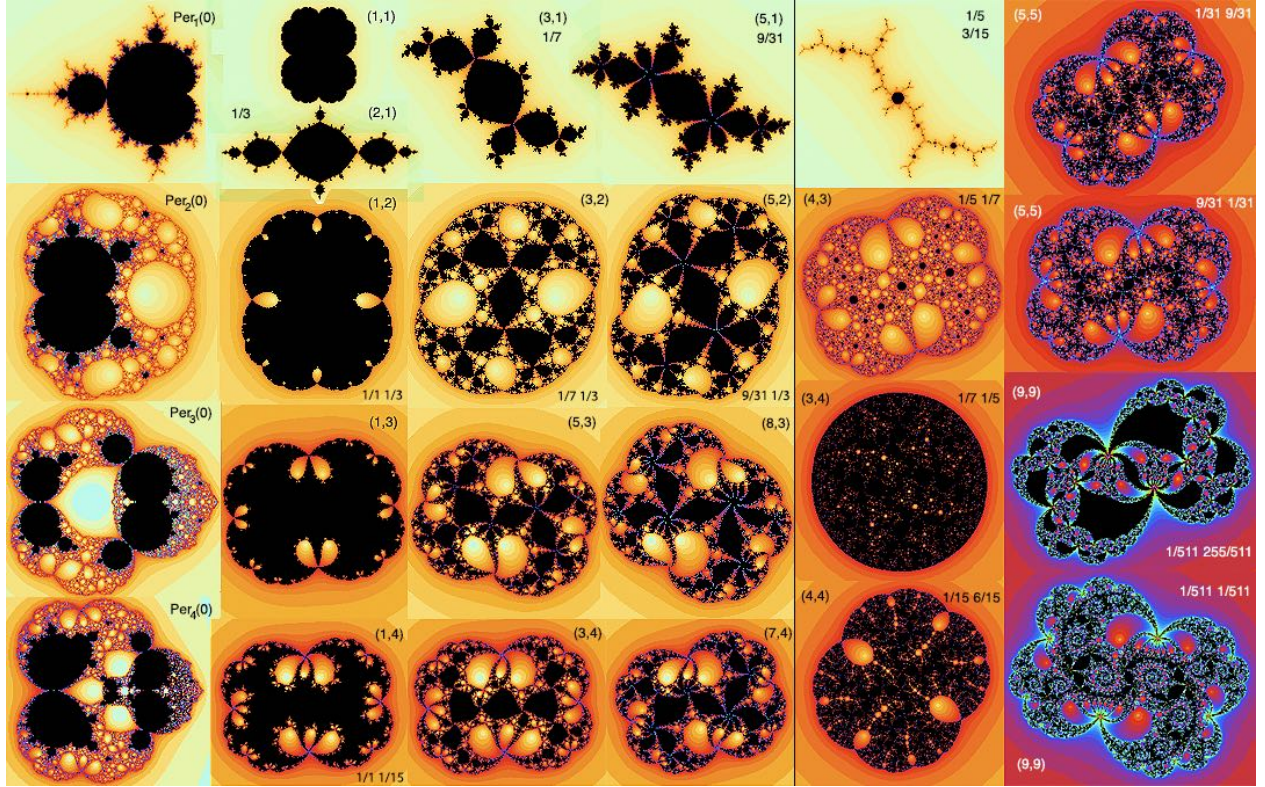


Fig 1: Left: Periodic moduli space slices and Julia set matings using rational functions (Devaney et al. 2013), which are also general for mating a global array of Julia sets with low period cases. Right: Medusa matings, complement those created by the slice method, mating Julia sets defined by any two external angles, using coefficients produced by the Medusa algorithm.

The $\text{Per}_k(0)$ generating functions are: $f_1(z) = z^2 + c$, $f_2(z) = c / (z^2 + 2z)$, $f_3(z) = (z-1)(z-c/(2-c)) / z^2$ and $f_4(z) = (z-4c/(10c+1))(z-(1+2c)/(1+6c)) / z^2$. Medusa matings are of the form $f(z) = (az^2 + 1 - a) / (bz^2 + 1 - b)$.

The Medusa method favours Julia sets identified by external angles as shown in fig 3, but the slice method can readily find matings with irrational flows, provided they are mated with low period attractors as in fig 2 right.

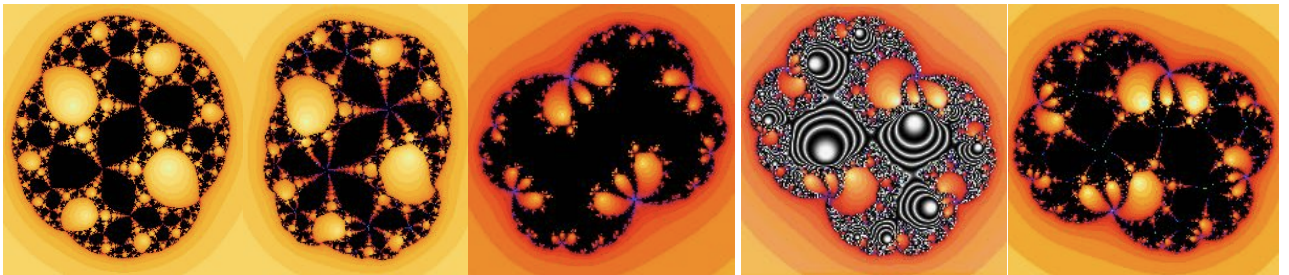


Fig 2: Left: Medusa $[1/7, 1/3]$ $[9/31, 1/3]$ and $[1/1, 1/15]$. Right: $\text{Per}_4(0)$ matings with a Siegel disc and a pd 5 parabolic set.

Although Medusa can diverge, or remain unstable for some values, it does give comparable results for many examples of periodic domains with odd external angles, where the methods appear to be homologous. For example (3,2) (5,2) and (1,4) above have homologous Medusa Julia matings $[1/7, 1/3]$ $[9/31, 1/3]$ and $[1/1, 1/15]$ shown in fig 2, implying the functions, while different, have conjugate dynamics. However, it is more challenging

to find correspondences in some other cases, although both approaches appear to give valid matings. Compare for example $[1/7 \ 1/15]$ in fig 3 with (3,4) above and the example from the smaller period 3 bulb fig 3 right, with location shown at (a). All three are topologically distinct matings.

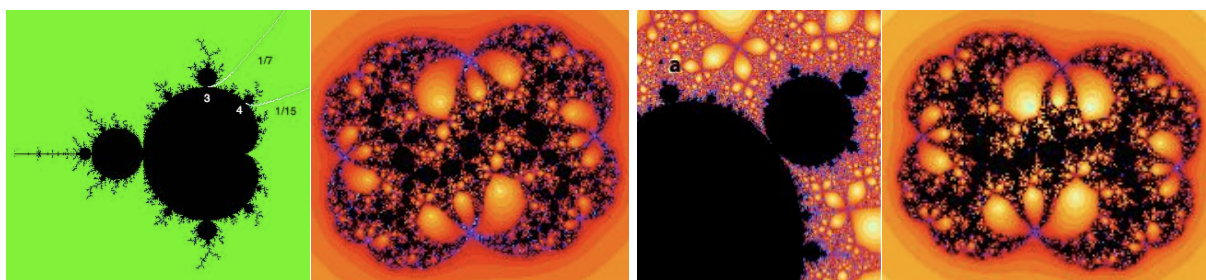


Fig 3: $[1/7 \ 1/15]$ mating periods 3 and 4 and the key $\text{Per}_4(0)$ period 3 mating at (a) indicates differences, which may be due to the generating function of the parameter plane $\text{Per}_4(0)$ since the right-hand image is asymmetric, as in figs 7, 8.

Neither does the $[1/7 \ 1/7]$ self-mating in fig 4 appear to be homologous to any of those of $\text{Per}_3(0)$, possibly due to its suppression of pd 3, although it does appear homologous to that of Arnaud Chéritat's Thurston algorithm.

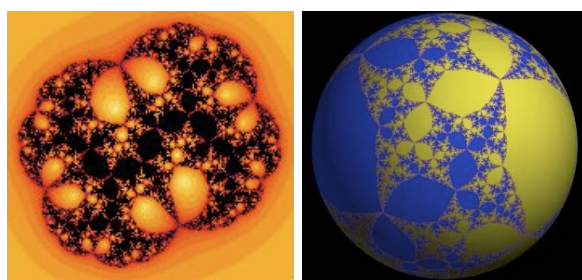


Fig 4: $[1/7 \ 1/7]$ which doesn't appear in $\text{Per}_3(0)$ and an equivalent mating by Arnaud Chéritat (2015).

Medusa correctly portrays both $[1/7, 1/5]$ and $[1/5, 1/7]$ mating the period 3 bulb to its period 4 dendritic Mandelbrot, as shown in fig 1, and it can portray two dendritic Mandelbrot satellite Julias on the same side of the x -axis as shown below for $[5/31 \ 1/5]$ and $[1/5 \ 5/31]$, shown below left and centre. But the $[1/5, 1/5]$ and $[5/31, 5/31]$ self-matings, shown at right have distinct appearances. Significantly the coefficients of $[1/5, 1/5]$ are complex conjugates. The second shows its structure to be a complementary fractal in the detail right.

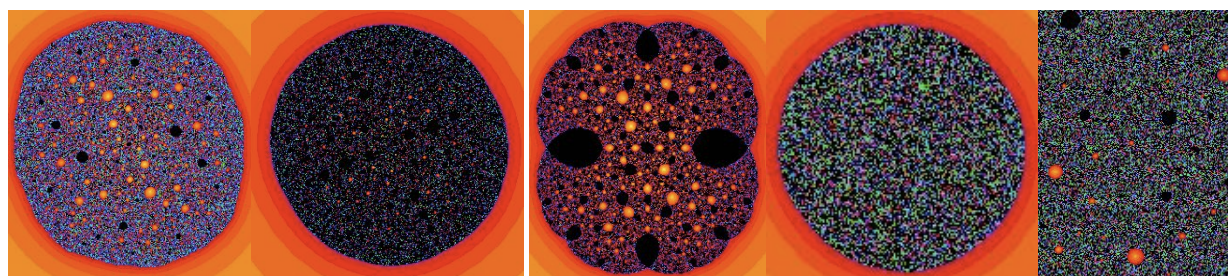


Fig 5: Medusa matings $[5/31 \ 1/5]$ and $[1/5 \ 5/31]$, and $[1/5, 1/5]$ and $[5/31, 5/31]$ main body between period bulb Julias and dendritic Mandelbrot Julia sets.

A situation where something provocative happens is the Medusa mating between the period 3 bulb Julia set (Rabbit) with the Julia set of the period 3 Mandelbrot on the negative x dendrite (Airplane). The Medusa algorithm for $[1/7, 3/7]$ and $[3/7, 1/7]$ don't look at face value like a mating between a bulb and a dendritic Mandelbrot Julia, as we saw in $[1/5, 1/7]$ and they are apparently homologous to one another as shown in fig 6.

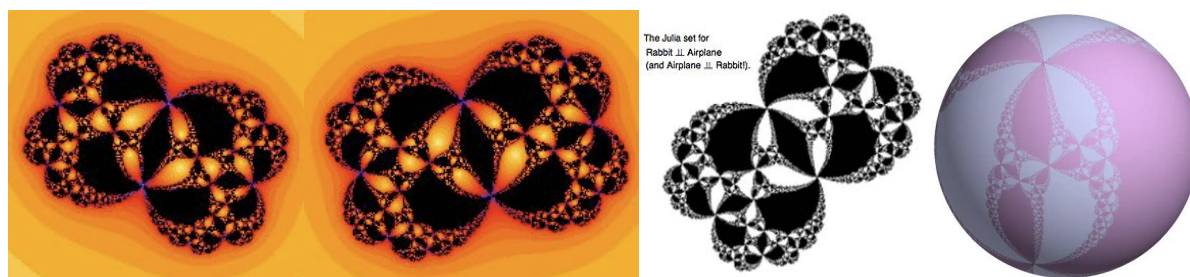


Fig 6: $[1/7, 3/7]$ and $[3/7, 1/7]$ compared with Tom Sharland's and Arnaud Chéritat's versions.

The same situation as in fig 6 applies also to $[1/511, 255/511]$ (fig 1), which is also a mating between a period 9 bulb and a period 9 dendritic Mandelbrot on the negative x-axis.

Wolf Jung has pointed out that these aspects can be explained by shared matings - 'different pairs of polynomials may give the same rational map. One of the simplest examples is that the mating of Rabbit and Airplane is the same as the mating of Airplane and Rabbit, up to a rescaling. In fact the map can be rescaled such that it is invariant under inversion $1/z$, although it is not a self-mating. Moreover, the fact that six Fatou components meet at a single point, can be explained in terms of ray connections'. This example is confirmed again the image right from Tom Sharland's (2012) Harvard lecture. Thurston equivalence means that Julia sets of matings are unique up to conjugacy classes via Mobius transformations.

In fig 7 we explore this mating using two versions of $\text{Per}_3(0)$. The period 3 dendritic Mandelbrot (a) has a mating looking as we would expect, in both Dark Heart (upper row) and Mandel (lower row) - very obviously the Airplane and Rabbit. The other period 3 regions in the parameter planes are (b) which is not homologous to fig 6 and (c), which differs in Dark Heart, but is identical to fig 6 in Mandel, raising a question about the relationship between them and whether the Julia sets form homologous matings under a Mobius transformation.

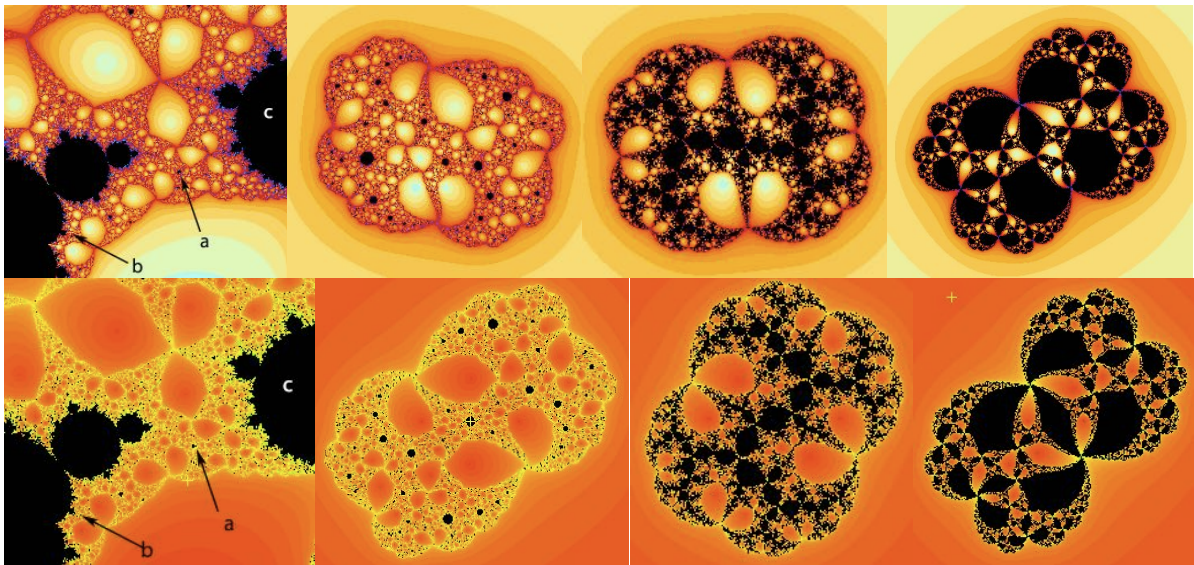


Fig 7: Dark Heart and Mandel versions of (3,3) $\text{Per}_3(0)$ matings show subtle differences of topology.

The two parameter planes illustrated in fig 8 appear at first sight to be identical but have subtle differences in their topology to the right of the central basin which ramifies into the Julia sets. The rational function in Mandel is $f_3^M(z) = (z^2 + c^3 - c - 1)/(z^2 - c^2)$ with period 3 critical orbit $\infty \rightarrow 1 \rightarrow -c$ and critical point 0, while the one in Dark Heart (Devaney et al. 2013) is $f_3^D(z) = (z-1)(z-c/(2-c))/z^2$ with $\infty \rightarrow 1 \rightarrow 0$ and critical point c . Both appear to give valid matings despite the asymmetry, so presumably must differ by a Mobius transformation.

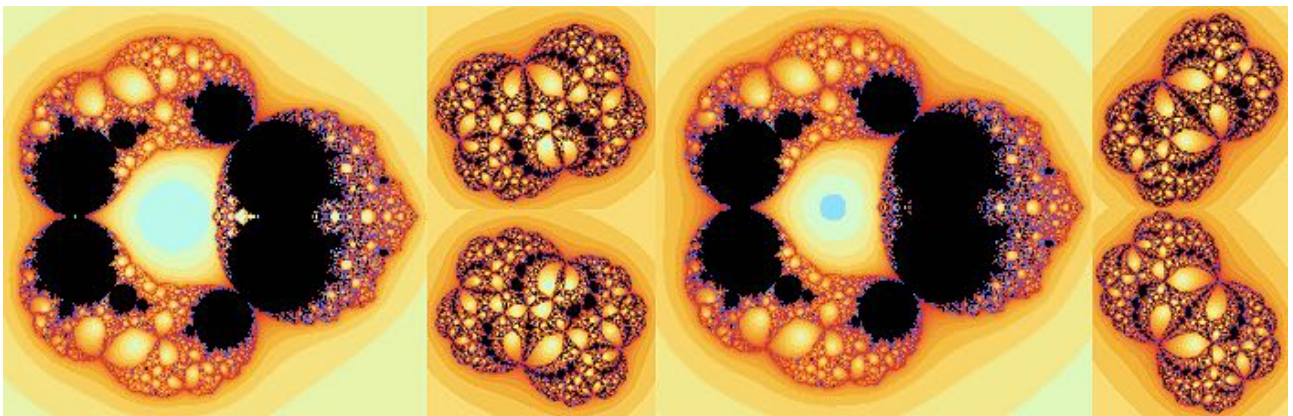


Fig 8: Running in Dark Heart, the two $\text{Per}_3(0)$ parameter planes and their Julia sets have subtle differences.

To seek a resolution for the period 4 case we need to generate a symmetric Julia spectrum by confining the zero and infinite critical points to zero and infinity as is the case for the period 3 version in Mandel:

$$f(z) = \frac{z^2 + p}{z^2 + q}, \infty \rightarrow 1 \rightarrow -c, \Rightarrow 1 + p = -c(1 + q), g = -c^2, p = -c(1 - c^2) - 1 = c^3 - c - 1, f(z) = (z^2 + c^3 - c - 1) / (z^2 - c^2)$$

We now need to assign an arbitrary point a to retain the correct degrees of freedom as shown below.

$$f(z) = \frac{z^2 + p}{z^2 + q}, \infty \rightarrow 1 \rightarrow a \rightarrow -c, 1 \rightarrow a \Rightarrow 1 + p = a(1 + q), a \rightarrow -c \Rightarrow a^2 + p = -c(a^2 + q), -c \rightarrow \infty \Rightarrow g = -c^2,$$

$$p = a(1 - c^2) - 1, a^2 + a(1 - c^2) - 1 = -ca^2 + c^3, a^2 + a((1 - c) - (c^2 - c + 1)) = 0, a = (c - 1) \pm (c - 1 \pm \sqrt{5c^2 - 6c + 5}) / 2,$$

$$f(z) = (z^2 + (c - 1 \pm \sqrt{5c^2 - 6c + 5})(1 - c^2) / 2 - 1) / (z^2 - c^2)$$

Because this generating function now involves a fractional power of c , the complex parameter plane becomes split, resulting in two “fermionic” parameter planes connected by the elliptic split illustrated (right) in fig 9 below. Comparison of these with the Medusa matings for the 6 period four locations in the quadratic Mandelbrot set (left) of these shows that the two planes provide a full representation of the matings, with all these cases and confirms the consistency of the two mating methods.

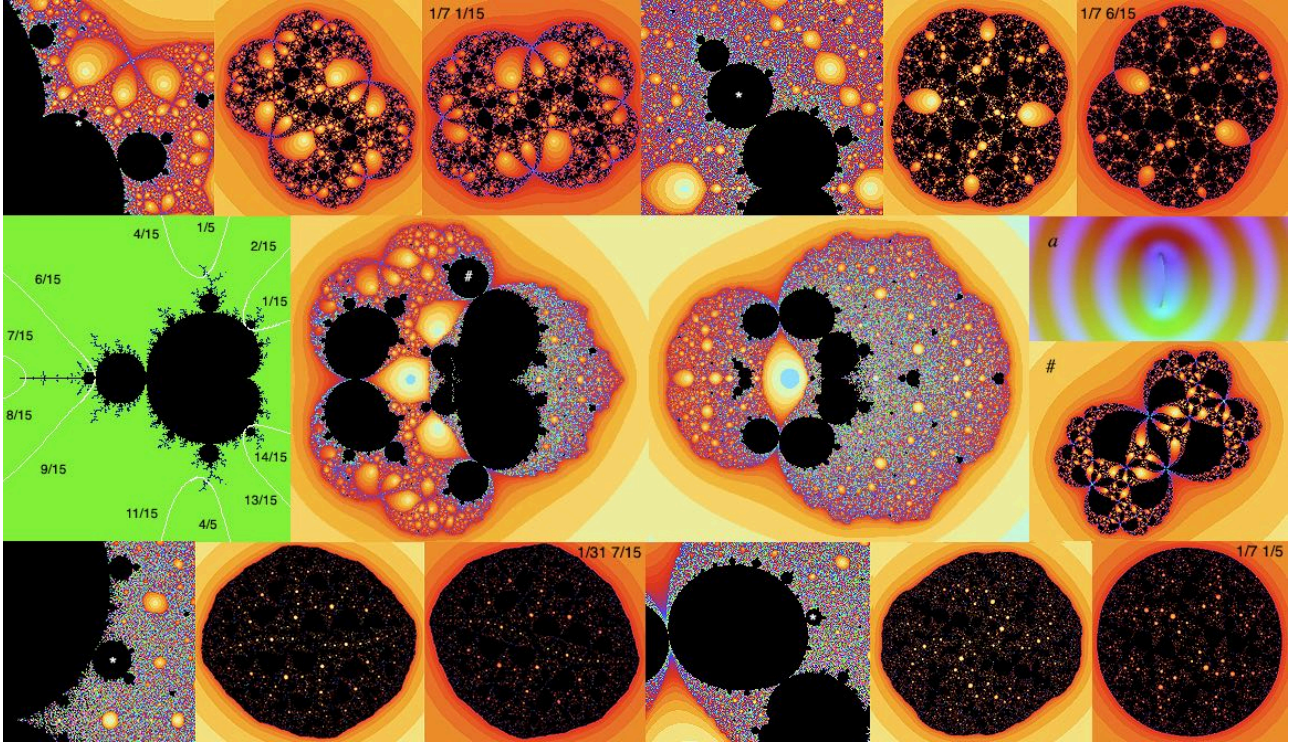


Fig 9: Global correspondence between Medusa matings for all the period 4 types and the “fermionic” Per4(0).

We now explore shared or equivalent matings further in Medusa. The two dendritic Mandelbrot matings $[3/7, 1/5]$ and $[1/5, 3/7]$ top row fig 10 appear to be equivalent to $[1/7, 6/15]$ and $[6/15, 1/7]$ on the period 3 and 2x2 bulbs, again suggesting shared matings.

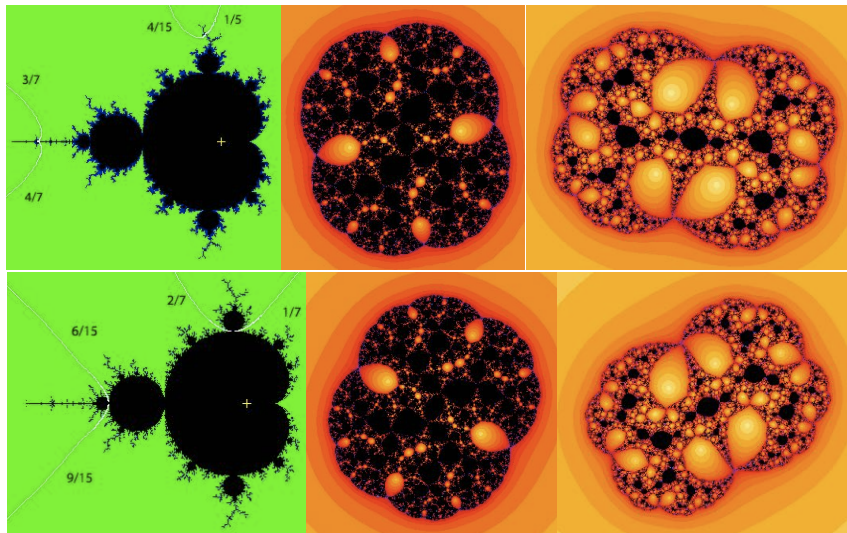


Fig 10: $[1/5 \ 3/7]$ gives the same Medusa mating as $[1/7 \ 6/15]$

Tom Sharland (2012) notes that the two matings in fig 11 are known to be equivalent. Indeed Medusa not only gives identical coefficients for both, but the inverse mating $[7/15 \ 1/5]$ is homologous to the original, even though the Fatou basins of zero (black) and infinity (shaded orange) have been exchanged.

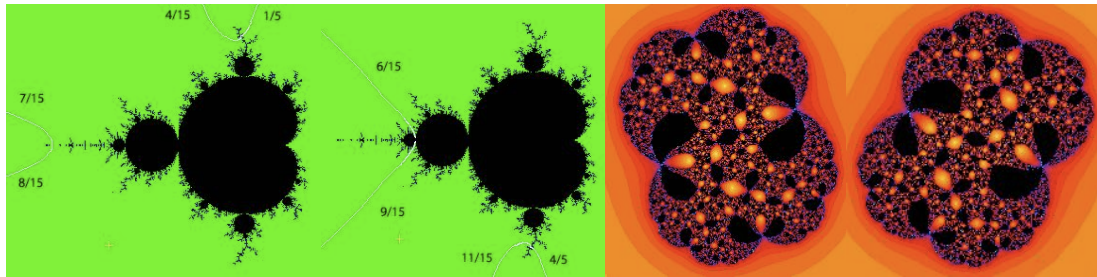


Fig 11: Equivalent matings $[1/5 \ 7/15]$ and $[4/5 \ 6/15]$ with their Julia set and that of the inverse matings.

Clustering is the condition where the critical orbit Fatou components group together to form a periodic cycle. Tom comments that the matings right in fig 11 all have period 3 cluster cycles with the same intrinsic data. But they certainly don't all look the same! In simple cases, (periods 1 & 2) the combinatorial data of a cluster completely defines a rational map, but in period 3 the experimental pictures suggest not.

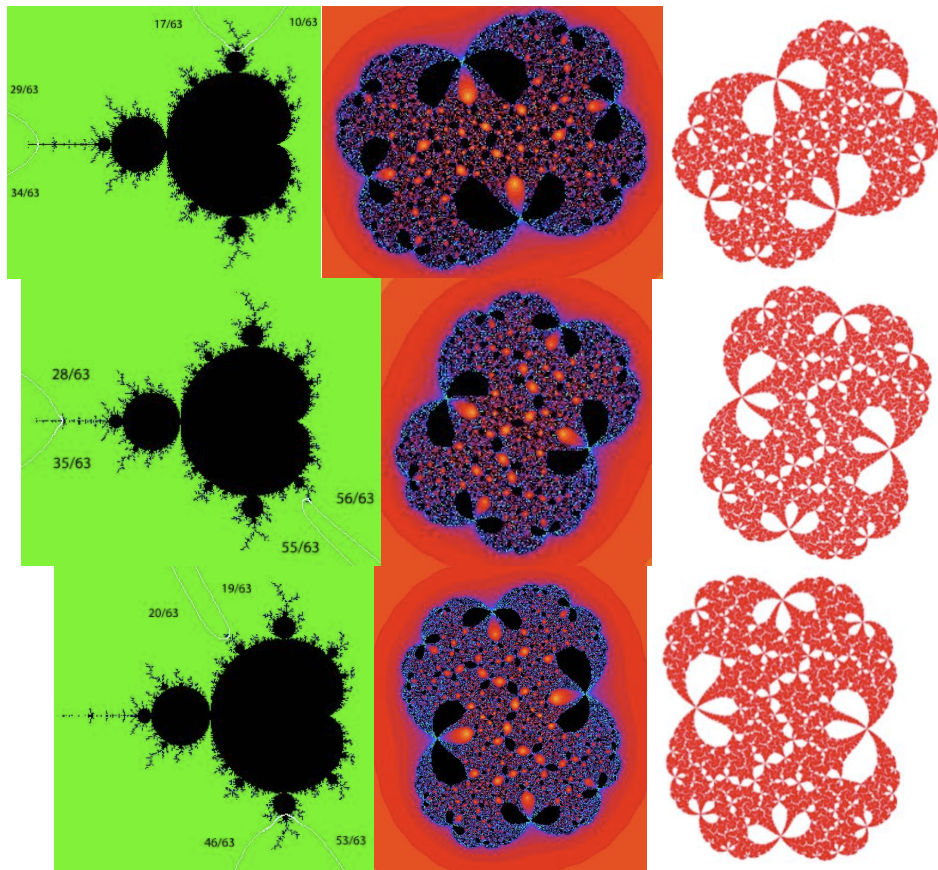


Fig 12: Equivalent period 3 cluster matings correspond to Tom Sharland's images, provided you pick the appropriate pair of ratios in the left-hand figures. Some appear to be topologically distinct.

Now let's turn to even denominators where we have rays to Misiurewicz points on the dendrites. There is no problem with the first denominator being even as $[1/4, 1/7]$ shows us below left, and $[1/4, 1/511]$ at right, but if we choose $[1/7, 1/4]$, we get the infinite Julia set shown centre. Noting that Medusa has placed the Julia set over infinity instead of zero - the correct thing to do as $[1/4]$ has no interior basin so it should sit on infinity, we can make the Mobius transformation $\frac{az^2 + 1 - a}{bz^2 + 1 - b} \rightarrow \frac{(1-b)z^2 + b}{(1-a)z^2 + a}$ and we have a nice Julia set whose coefficients are distinct from those of $[1/4, 1/7]$ which is otherwise homologous to $[1/4, 1/7]$.

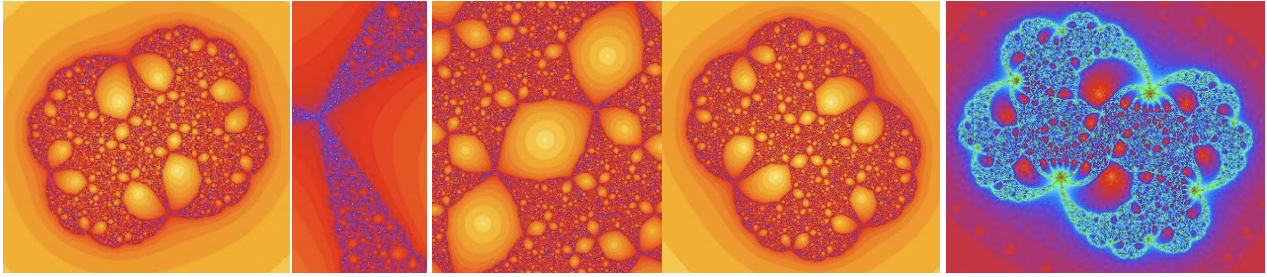


Fig 13: Left: $[1/6, 1/7]$ with dendritic tree detail. Centre: $[1/7, 1/6]$ and its Möbius inversion. Right: $[1/4, 1/511]$

$[1/4, 1/6]$ and $[1/4, 1/2]$ also look to be plausible because they are mating a chaotic dendritic Julia set to another one, so the whole plane is close to Julia ... but is this the case if one should have complimentary shading?

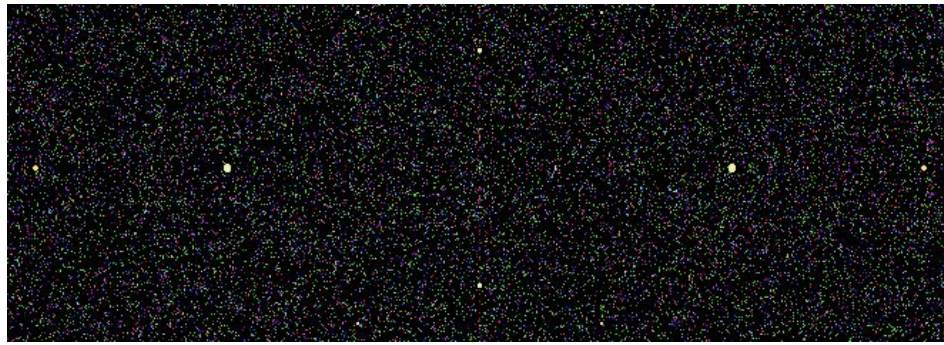


Fig 14: Starry sky with symmetries. Medusa mating of two dendritic Julia sets $[1/4, 1/2]$

Arnaud Chéritat's Thurston algorithm does give clear evolutionary portraits of matings of dendritic Julia sets including $[1/6, 5/14]$, in fig 15. He notes that according to Shishikura and Milnor, this gives a Lattès map.



Fig 15: Three stages in Arnaud Chéritat's (2015) movie of dendritic Julia set mating $[1/6, 5/14]$ appears to solve this. Medusa iterations remained unstable for these values. Many more available at Arnaud's link below.

References

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