



Fig 7c: Dynamics of the function  $f_c(z) = \frac{z^2 - c^3}{z^2 + c}$  with both finite and infinite critical points.

The function  $f_c(z) = \frac{z^2 - c^3}{z^2 + c}$  in fig 7c demonstrates the dynamics when we have to take account of both the critical points with zero derivative and those with an infinite derivative. Here we have

$f'_c(z) = 2(c - c^3)z / (z^2 + c)^2$ ,  $f'_c(0) = 0$ ,  $f'_c(\pm(-c)^{1/2}) = \infty$  and since  $\pm(-c)^{1/2} \rightarrow \infty \rightarrow 1$ , we can effectively use 0 and 1 as the critical points. Moreover, we can see by swapping 0 and  $\infty$  by  $z \leftrightarrow c^2 / z$  in both the

domain and range that  $c^2 / f_c(c^2 / z) = c^2 \frac{(c^2 / z)^2 + c}{(c^2 / z)^2 - c^3} = \frac{c^6 + c^3 z^2}{c^4 - c^3 z^2} = \frac{z^2 + c^3}{z^2 - c} = f_{-c}(z)$ , so the parameter

plane of  $\infty$  is simply the parameter plane of 0 flipped in the  $c$  direction. This results in the combined plane shown in (1), with the two individual planes in (3,4). The combined plane forms a symmetrical figure 8 with pairs of bulbs running in two series (2) in the interior of the black chaotic region, which consists of points which fail to head to any attractor from either critical point.

Close inspection however reveals that, the two, apparently symmetrical sides differ. Superimposing the left and right for the period-16 bulb (2i) shows that, although these overlap, there are key differences. The expanded superimposed period 13 and 16 bulbs (5,6) have clear patterns indicating critical point interference. Intriguingly for the period 13 and 16 cases, the bulb of 0 intersects both with regions of identical asymptotic period in the bulb of  $\infty$ , and with  $c$  values where  $\infty$  is pre-periodic, or chaotic, and hence in the Julia set. Note that the two critical points differ fundamentally in their behavior, because one is a zero of the derivative, which is not a solution of the numerator and the other is simply a solution of the denominator. This means that the Julia sets on either side have distinct dynamics. See: <http://dhushara.com/DarkHeart/index.htm> for full details.